Equilibria in Bottleneck Games

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Abstract

This paper introduces a bottleneck game with nite sets of commuters and departing time slots as an extension of congestion games by Milchtaich (1996). After characterizing Nash equilibrium of the game, we provide suf- cient conditions for which the equivalence between Nash and strong equilibria holds. Somewhat surprisingly, unlike in congestion games, a Nash equilibrium in pure strategies may often fail to exist, even when players are homogeneous. In contrast, when there is a continuum of atomless players, the existence of a Nash equilibrium and the equivalence between the set of Nash and strong equilibria hold as in congestion games (Konishi, Le Breton, and Weber, 1997a).

1 Introduction

A bottleneck model is used in analyzing tra c congestion during rush hours, such as Smith (1983), Daganzo (1985) and Arnott et al. (1990), introduce some heterogeneity of commuters.

In this paper, we define a bottleneck game with a finite set of departure time slots. Each commuter has preferences on two arguments: her departure time and the length of the queue in which she has to wait to pass through the bottleneck. Our game is an anonymous game with congestion generated by a queue structure without imposing a specific form of trip costs function. In this sense, our model can be regarded as an abstract generalization of bottleneck models in the aforementioned papers. Moreover, this abstract setup allows us to interpret our model in a different context other than traffic congestion. For example, consider a location choice problem along a river, in which residents pollute the river while the river has an ability to abate pollution up to some level (capacity) at each location of the river. We can allow residents' arbitrary preferences over locations (such as scenic and/or convenient locations) on the river, resulting in emergence of congested locations causing pollutions for downstream locations.

Mathematically, our model is also an extension of the congestion game by Milchtaich (1996), which has following three properties.[1] Anonymity (A), Congestion (C) and Independence of Irrelevant Choices (IIC). First, A requires that the payoff of each player depends on the number of players who choose each action and not on the players' names. Second, C states that the payoff of each player increases if another player who had chosen the same strategy chooses a different strategy. Finally, IIC states that the payoff of a player is not affected even if another player that chooses a different strategy from hers switches to another strategy that is also a different strategy from hers. In this game, Milchtaich (1996) shows that a congestion game always has a Nash equilibrium in pure strategies. Konishi et al. (1997a) shows that in the same model, any strictly improving coalitional deviation from a Nash equilibrium results in another Nash equilibrium, thus implying a congestion game also admits a strong equilibrium that is immune to any strictly improving coalitional deviation. They also show that if there is a continuum of atomless players, then the sets of Nash and strong equilibria coincide with each other.

Our bottleneck game does not satisfy IIC, whereas the other two conditions hold (though C applies in a strict sense only after a queue forms by exceeding the capacity). Specifically, IIC would be violated in the case where a player who had departed later then switched to an earlier departure time and thereby possibly creating a longer queue for some of those players which she leaps over. With this

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[1] The name "congestion game" is sometimes attributed to a class of games introduced by Rosenthal (1973), who considers a situation in which players choose a combination of primary factors out of a certain number of alternatives. Each player's payoff is determined by the sum of the costs of each primary factor she chooses, while the cost of each primary factor depends on the number of players who choose it, and not on the players' names. Rosenthal (1973) proved that there always exists at least one pure-strategy Nash equilibrium by constructing a potential function, which is later formalized by Monderer and Shapley (1996). However, these

di erence, we rst show that the equivalence between Nash and strong equilibria under some conditions (Propositions 2, 3, and 4), show that a Nash equilibrium may not exist even when players are Homogeneous (H) and other stringent conditions such as Single-Peakedness (SP) and Order-Preservation (OP) on the payo function are satis ed (Examples 4 and 5). With an even more stringent condition, we show the existence of Nash equilibrium (Proposition 5). These results are in stark contrast with the ones in Milchtaich's congestion games: Nash equilibrium always exists, and it is hard to ensure the equivalence between Nash and strong equilibrium due to coordination failures unless players are homogeneous. In contrast, when players are atomless, we can establish both the existence of Nash equilibrium and equivalence between Nash and strong equilibria exactly in the same way as in congestion games (Proposition 6).

The rest of the paper is organized as follows. In Section 2, we de ne our bottleneck game with a nite number of players. In Section 3, we provide three su cient conditions under which Nash and strong equilibria are equivalent to each other. In Section 4, we show that our bottleneck game may not have a Nash equilibrium in pure strategies even when players are homogeneous. We also provide a positive result for the existence although the conditions are very stringent. Section 5 introduces a bottleneck game with atomless players, and we show that the existence of Nash and the equivalence between Nash and strong equilibria all hold in this idealized environment. Section 6 concludes.

2 The Model with a Finite Number of Players

We consider a commuting road with a nite number of departing time slots. Let $t = 1, \ldots, T$ be available departing time slots ($t = 1$ is the earliest). Each discrete time unit can represent every minute or every ve minutes, for example. Let the set of departing time slots be $T = \{1, \ldots, T\}$. Let q_{t-1} be the length of the resulting queue at departing time slot $t - 1$. Then, the length of the queue at time slot t can be calculated as $q_t = \max\{0, q_{t-1} + m_t - \bar{q}\}$ ... $q_i; q$

Note that although $q = 0$ holds irrespective of $\tilde{q} = 0$ or $\bar{q} <$

3 Equivalence between Nash and Strong Equilibria

A *coalitional deviation* from   is a pair of $(C; \hat{\ }^{C})$ such that (i) $C \neq \emptyset$, and (ii) for all $i \in C$, $u^{i}(\hat{\ }) > u^{i}(\ )$, where $\hat{\ } = (\hat{\ }^{C};\ ^{-C})$. A *strong equilibrium* is a strategy profile such that there is no coalitional deviation from . In a special case, we can show that Nash equilibrium is unique and is equivalent to strong equilibrium. This is a unique result in our domain, since in the domain of Konishi et al. (1997a), it is virtually impossible to exclude coordination failure: that is, it is not easy to show the equivalence between Nash and strong equilibria.

Proposition 2. Suppose that there is a Nash equilibrium   with a unique connected terrace {134()115562

Claim 2. Suppose that is a Nash equilibrium, and that $(C; \hat{}_C)$ is a coalitional deviation from.

The above result relies both on the uniqueness of connected terrace and the absence of single terraces in equilibrium. The next example shows that the equivalence result may not hold if the conditions are not satis ed.

Example 2. Let N = f 1; 2; 3; 4; 5; 6g and T = f 1; 2; 3; 4; 5g with capacity $c = 1$. Players 1, 2, 3 and 4 are attached to time slots 1, 2, 4, and 5, respectively. Players 5 and 6 have the following preferences, respectively:

$$u^5(1;0) > u^5(2;0) > u^5(4;0) > u^5(5;0) > u^5(1;1) > u^5(2;1) > u^5(4;1) > u^5(5;1) > others$$

$$u^6(4;0) > u^6(5;0) > u^6(1;0) > u^6(2;0) > u^6(4;1) > u^6(5;1) > u^6(1;1) > u^6(2;1) > others$$

There are two Nash equilibria: = (1; 2; 4; 5; 1; 4) and $'$ = (1; 2; 4; 5; 4; 1). In these cases q_3 = 0. Only is a strong equilibrium. □

An additional natural condition allows Proposition 2 to extend to the case with multiple connected terraces. We say that the time slot t_i^* 2 T is an optimal slot for player i 2 N if $u^i(t_i^*; 0) > u^i(t; 0)$ for all t 2 T ; t 6 t_i^*.

Single-Peakedness (SP). Let player i's optimal slot be t_i^* 2 T . Then, for all i 2 N, and all $t' < t < t \quad t^* < t <$

There is a Nash equilibrium $= (1, 1, 1, 3, 3, 2, 3)$, but $(C, \hat{}_C) = (\{6, 7\}, (3, 2))$

k choose

] ;

Again, this contradicts Claim 3. □

Proof of Proposition 4. Suppose that is a Nash equilibrium, and that $(C; \hat{}_C)$ is a coalitional deviation from . We will derive a contradiction.

Step 1. Find t 2 T such that q (^) < q ( ). If there exist multiple such slots, take the earliest one. Denote by $[\underline{t}; \bar{t}]$ the connected terrace where t belongs. Note that some player i 2 C switches to $\hat{}_i$ 62 $[\underline{t}; \bar{t}]$ at ^.

Step 2. Find a player who deviates to slots in $[\underline{t}; \bar{t}]$ at ^.
By Claim 5, there must be at least one such player. Among these players, let the player who chooses the latest slot at ^ be player j 2 C. Note that player j chooses $_j$ at which does not belong to $[\underline{t}; \bar{t}]$, say $[\underline{t}'; \bar{t}']$. That is, player j chooses $_j$ 2 $[\underline{t}'; \bar{t}']$ at and $\hat{}_j$ 2 $[\underline{t}; \bar{t}]$ at ^.

Step 3. Find a player who deviates to slots in $[\underline{t}'; \bar{t}']$ at ^, and name player k the one among such players who chooses the latest slot at ^
Likewise in Step 2, such player must be found due to player j's deviation from $[\underline{t}'; \bar{t}']$. Let player k choose 2 $[\underline{t}'';$

Note that by H,

$$u^{i(l+1)}(\hat{}_{i(l)}; q_{\hat{}_{i(l)}}(\hat{})) = u^{i(l)}(\hat{}_{i(l)}; q_{\hat{}_{i(l)}}(\hat{})): \qquad (6)$$

Hence, from (3), (4), (5) and (6), we obtain

$$\begin{aligned} u^{i(l+1)}(\ _{i(l+1)}; q_{\ i(l+1)}(\ )) &> u^{i(l+1)}(\hat{}_{i(l)}; q_{\hat{}_{i(l)}}(\hat{}_{i(l)}; \ _{i(l+1)})) \\ & \quad u^{i(l+1)}(\hat{}_{i(l)}; q_{\hat{}_{i(l)}}(\hat{})) \\ &= u^{i(l)}(\hat{}_{i(l)}; q_{\hat{}_{i(l)}}(\hat{})) \\ &> u^{i(l)}(\ _{i(l)}; q_{\ i(l)}(\ )): \end{aligned}$$

However, this yields a cycle on the preference:

$$\begin{aligned} u^{i(1)}(\ _{i(1)}; q_{\ i(1)}(\ )) &< u^{i(1)}(\hat{}_{i(1)}; q_{\hat{}_{i(1)}}(\hat{})) \\ &< u^{i(2)}(\ _{i(2)}; q_{\ i(2)}(\ )) \\ &< u^{i(2)}(\hat{}_{i(2)}; q_{\hat{}_{i(2)}}(\hat{})) \\ & \ \ \vdots \\ &< u^{i(k)}(\ _{i(k)}; q_{\ i(k)}(\ )) \\ &< u^{i(k)}(\hat{}_{i(k)}; q_{\hat{}_{i(k)}}(\hat{})) \\ &< u^{i(k+1)}(\ _{i(k+1)}; q_{\ i(k+1)}(\ )) \\ &= u^{i(1)}(\ _{i(1)}; q_{\ i(1)}(\ )); \end{aligned}$$

which is a contradiction.

4 (Non)existence of Nash Equilibrium

Unfortunately, even under homogeneity, the existence of Nash equilibrium is not guaranteed. In fact, the following simple example shows that there may not be a Nash equilibrium even under H together with SP and another stringent condition, Order Preservation (OP) introduced by Konishi et al. (1997b) that investigates positive externality games (see below).

Order Preservation (OP). For all i 2 N, all $t; t^0$ 2 T and all $k; k^0$ 2 Z_+,

$$u^i(t;k) \quad u^i(t^0;k^0) \ () \quad u^i(t;k+1) \quad u^i(t^0;k^0+1):$$

The following Boundedness (B) condition together with OP enables us a tractable representation of payo functions.

Boundedness (B). Suppose that C holds. For all $t; t^0$ 2 T with $u^i(t;0) <$ $u^i(t^0;0)$ there exists k_{tt^0} 2 Z_+ such that $u^i(t;0) > u^i(t^0;k_{tt^0})$.

The following result is a variation of the result in Konishi and Fishburn (1996).[3]

Fact. Under A, B, C, and OP, utility function u^i has a quasi-linear representation. There is a vector $v^i = (v^i(1); :::; v^i(T))$ 2 $\mathbb{R}^{\mathcal{T}}$ such that for all $t; t'$ 2 T , and all $k; k'$ 2 $\mathbb{Z}_+$,

$$u^i(t;k) \quad u^i(t';k') \Leftrightarrow \quad v^i(t) \quad k \quad v^i(t') \quad k':$$

Example 4. Consider the following three-player, three-time-slot game with A, B, C, H, OP, and SP (capacity c = 1): $v(1) > v(2) > v(1) \quad 1 > v(3) > v(2) \quad 1 > v(1) \quad 2 > :::$ Then, there is no pure strategy equilibrium. To see this, rst note at least one player chooses 1 in a Nash equilibrium. Let player 1 be such a player. Without loss of generality, player 2 weakly earlier departure time than player 3. There are ve cases: (i) (1; 1; 1) then a player moves to 3, (ii) (1 ; 1; 2) then player 3 moves to 3, (iii) (1; 1; 3) then player 1 or 2 moves to 2, (iv) (1 ; 2; 2) then player 2 or 3 moves to 3, and (v) (1; 2; 3) then player 3 moves to 1. Thus, there is no Nash equilibrium in pure strategy. □

Therefore we seek a stronger concept, which we call symmetric single-peakedness (SSP). Symmetric single-peakedness re ects a player who values the trade-o between departing at her optimal slot and the queue-length at a one-to-one ratio. That is, departing k slots later (earlier) than the optimal slot is equivalent to facing an added queue-length of k at her optimal slot. Formally,

Symmetric single-peakedness (SSP). For all i 2 N, let t_i^* 2 T be an optimal slot. Player i's pay(ednesu9727(vd[(6 atisre)-27(v)27[(,)]TJ /F12 9.91.861f -4.98710.516 0 Td[(v)]TJ

Step 1 Set $n' = n$.

Step 2 At slot t^*, put $(c+1)$ players whenever possible, and proceed to Step 3. If $n' < c+1$, put all n' players at slot t^*, and terminate.

Step 3 Update n' with $n' \quad (c \quad 1)$, i.e., $n' ! \quad n' \quad (c \quad 1)$.

Step 4 Set $= 1$.

Step 5 While $t^* \qquad > 0$ and $n' > 0$:

Step 5-1 At slot ($0 < d$

At this pro le the queue-length vector $q(\ )$ becomes

$$
\begin{aligned}
q(\ ) &= (q_1; :::; q_{\ -1}; q_{\ }; q_{\ +1}; :::; q_{\ }; q_{\ +1}; :::; q_{\ -1=2 \ -}; q_{\ }; :::) \\
&= (0; :::; 0; 1; 2; :::; t^* \ \ t_1 + 1; t^* \ \ t_1; :::; 1; 0; :::): \qquad (7)
\end{aligned}
$$

SSP and OP imply

$$
\begin{aligned}
u(t_1 \ \ 1; 0) = u(t_1; 1) = \ \ &= u(t^*; t^* \ \ t_1 + 1) \\
&= u(t^* + 1; t^* \ \ t_1) = \ \ = u(t_2 \ \ 1; 1) = u(t_2; 0):
\end{aligned}
$$

First, note that player i with $\ _i$ 2 $[t_1; t_2]$ cannot improve by departing later in $[t_1; t_2]$, since the queue-length at switched slot, $\ '_i$ is the same as in (7), so player i is indi erent between $\ _i$ and $\ '_i$.

In addition, these players cannot improve by departing earlier in $[t_1; t_2]$, since the queue-length at switched slot, $\ '_i$, compared to (7), increases by one, so they are worse o by switching to $\ '_i$.

Next we consider the case when they depart later or earlier out of the connected terrace. At $\ '_i$, they face a queue of length zero or one if $\ '_i = t_1 \ \ 1$ or of length zero otherwise. If $\ '_i = t_1 \ \ 1$ and $q_{\ -1}(\ '_i; \ _{-i}) = 0$, player i is indi erent between $\ '_i = t_1 \ \ 1$ and $\ _i$. If $\ '_i = t_1 \ \ 1$ and $q_{\ -1}(\ '_i; \ _{-i}) = 1$, $\ '_i = t_1 \ \ 1$ is worse than $\ _i$. If $\ '_i \neq t_1 \ \ 1$, $u(\ '_i; 0) < u(t_1 \ \ 1; 0) = u(t_2; 0)$, they making worse-o .

Player i in slot $t_1 \ \ 1$, if any, does not depart earlier than slot $t_1 \ \ 1$ or later than slot t_2 by the same logic in the above. Player i

In this case, using a similar argument as in case (A)-(i), no player has an incentive to switch their slots.

(B) Suppose $t^* = 1$. This is a variant of the case (A)-(ii), and it is shown that no player has an incentive to switch their slots. □

Any property imposed on Proposition 5 seems required for a Nash equilibrium to exist. Indeed, once OP is dropped, then the existence of Nash equilibria may not be guaranteed any more as the following example shows.

Example 4. Let $N = \{1, 2, 3, 4\}$ and $T = \{1, 2, 3, 4\}$ with capacity $c = 1$. Players have the following preferences.

$$u(2;0) > u(1;0) = u(2;1) = u(3;0) > u(1;1) > u(3;1) > u(2;2) = u(4;0) > others.$$

In this example, H and SSP with optimal time slot $t^* = 2$ are satisfied, while OP is not, since $u(2;0) > u(1;0)$ but $u(2;1) = u(1;0) > u(1;1)$. Then, this example does not admit any pure strategy Nash equilibrium. To see, first consider four cases: (i) $(1;2;2;3)$ then player 4 moves to 1. (ii) $(1;2;2;1)$ then player 3 moves to 3. (iii) $(1;2;3;1)$ then player 4 moves to 2. (iv) $(1;2;3;2)$ then player 3 moves

$t \in \mathcal{T}$. Note that we can define $\tilde{q}(\sigma)$ and $q(\sigma)$ exactly in the same way as before: $\tilde{q}_\tau(\sigma) = q_{\tau-1}(\sigma) + \mu(\sigma^{-1}(\tau)) - c$, and $q_\tau(\sigma) = \max\{\tilde{q}_\tau(\sigma); 0\}$. By A, the payoff function $u^i(t;\sigma)$ can also be written as $u^i(t;\sigma) = v^i(t; q_t(\sigma))$.

Under the atomless player assumption, we will assume Schmeidler's technical assumption.

Regularity (R) (Schmeidler, 1973). (i) For all $i \in I$, and all $t \in \mathcal{T}$, $u^i(t;\sigma)$ is continuous. Thus, all utility functions are uniformly bounded and there exists a positive constant K such that $|u^i(t;\sigma)| < K$ for all $i \in I$, $t \in \mathcal{T}$, and σ. (ii) For all σ and all $t, t' \in \mathcal{T}$, the set $\{i \in I : u^i(t;\sigma) > u^i(t';\sigma)\}$ is measurable.

Proposition (Schmeidler, 1973). Under A and R, there exists a Nash equilibrium in pure strategies.

A strategy profile σ is a **strong equilibrium** if there is no measurable subset $C \subseteq I$ with $\mu(C) > 0$ and a strategy profile $\hat{\sigma}$ of players in C such that $u^i(\hat{\sigma}_i;\hat{\sigma}) > u^i(\sigma_i;\sigma)$ almost everywhere on C, where $\hat{\sigma} = ((\hat{\sigma}_i)_{i \in C}; (\sigma_i)_{i \notin C})$. We will impose the following congestion condition.

Congestion (C) $v^i(t; q)$ is strictly decreasing in q for all $t \in \mathcal{T}$ and all $q \in \mathbb{R}_+$.

The main result of this section is:

Proposition 6. Consider an atomless game. Under A, C, and R, the sets of Nash and strong equilibria coincide with each other.

Proof. Suppose that σ is a Nash equilibrium while it is not a strong equilibrium. Then, there exist a coalition C with $\mu(C) > 0$ and a strategy profile $\hat{\sigma}$ for C such that $u^i(\hat{\sigma}_i;\hat{\sigma}) > u^i(\sigma_i;\sigma)$, where $\hat{\sigma} = ((\hat{\sigma}_i)_{i \in C}; (\sigma_i)_{i \notin C})$. Note that $\hat{\sigma}_i \in \{t' \in \mathcal{T} : q_{t'}(\sigma) \leq 0\}$ holds for all $i \in C$. It is because player i would have moved under strategy profile σ, contradicting σ's being a Nash equilibrium, otherwise. Thus, $\{t' \in \mathcal{T} : q_{t'}(\sigma) > 0\} \subseteq \{t' \in \mathcal{T} : q_{t'}(\hat{\sigma}) > 0\}$.

Assume now that there is a time slot $t \in \{t' \in \mathcal{T} : q_{t'}(\hat{\sigma}) > 0\}$ with $q_t(\hat{\sigma}) > q_t(\sigma)$. Take the earliest time slot of this kind t. Then, $C \cap \{i' \in N : \hat{\sigma}_{i'} = t\} \neq \emptyset$. Let i be such a player. Since σ is a Nash equilibrium, $v^i(\sigma_i; q_{\tau_i}(\sigma)) \geq v^i(t; q_t(\hat{\sigma}))$ must hold. This is a contradiction with C's being profitable deviation. Thus, for all $t \in \{t' \in \mathcal{T} : q_{t'}(\hat{\sigma}) > 0\}$, $q_t(\hat{\sigma}) \leq q_t(\sigma)$ holds. Since $\{t' \in \mathcal{T} : \tilde{q}_{t'}(\sigma) > 0\} \subseteq \{t' \in \mathcal{T} : \tilde{q}_{t'}(\hat{\sigma}) > 0\}$, $q_t(\hat{\sigma}) = q_t(\sigma)$ holds for all $t \in \{t' \in \mathcal{T} : q_{t'}(\sigma) > 0\} = \{t' \in \mathcal{T} : q_{t'}(\hat{\sigma}) > 0\}$. Hence, a deviation C with $\hat{\sigma}$ cannot improve on Nash equilibrium σ. This implies from each other in the nite case. Somewhat surprisingly, the presence/absence of single-terraces (time slots that are chosen by the same number of players as the capacities) can alter the structure of the equilibria of the bottleneck game. This is because there is an asymmetry between an increase and a reduction in population at single-terraces: the former reduces payo s while the latter has no e ect on them. In contrast, in an atomless bottleneck game, we need essentially no condition for the result. There is no such asymmetry: players can simply choose the most preferable time slot given the queue structure without a ecting the queues. This is why we can recover the nice equivalence result between Nash and strong equilibria as in Konishi et al. (1997a).

Thus, whether the tra c bottleneck model started by Vickrey (1969) would provide us useful insights or not depends on how we interpret the "atomless" assumption of the model. If we accept this assumption as an reasonable approximation of the real world, we can enjoy nice properties and rich results of the model. However, if we question the legitimacy of atomless players, then we need to su er from the ill-behaved model coming from nite problems.

References

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