Equilibrium Player Choices in Team Contests with Multiple Pairwise Battles*

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Abstract

We consider games in which team leaders strategically choose the order of players sent to the battlefield in majoritarian team contests with multiple pairwise battles as in Fu, Lu, and Pan (2015 American Economic Review). We consider one-shot order-choice games and battle-by-battle sequential player choice games. We show that as long as the number of players on each team is the same as the number of battles, the equilibrium winning probability of a team and the ex ante expected effort of each player in a multi-battle contest are independent of whether players' assignments are one-shot or battle-by-battle sequential. This equilibrium winning probability and ex ante expected total effort coincide with those where the player matching is chosen totally randomly with an equal probability lottery by the contest organizer. Finally, we show how player choices add subtleties to the equivalence result by examples.

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1 Introduction

In their influential paper on group contests, Fu, Lu, and Pan (2015) analyze a multi-battle team contest in which players from two rival teams form pairwise matches to compete in ton and Romano (1998) under the assumption that each individual match has an exogenously fixed winning probability. Interestingly, they show that there is a mixed strategy equilibrium in which both teams assign the same probability to every ordering of the players, and that the expected winning probability is unique (von Neumann's minimax theorem in a two-person winning probability of a team in the totally mixed equilibrium in Hamilton and Romano (1998) is the same as that where the contest organizer chooses a matching of players totally randomly (Proposition 1). Using this result, we show that the totally mixed strategy Nash equilibrium in Hamilton and Romano (1998) extends to a one-shot order choice game in the Fu, Lu, and Pan multi-battle contest environment in which each player's effort level is endogenously determined, and that the expected winning probability of a team is the same when the contest organizer chooses a matching of players totally randomly (Theorem 1). Although Fu, Lu, and Pan (2015) assume that the pairwise player matching in their multi-battle contests is fixed, we show that their invariance result regarding the outcome (winning probability) of each pairwise battle is more general than that—as long as a pair of players are matched in one of the multiple battles in a team contest, the expected outcome (winning probability) stays the same, irrespective of the rest of the matches. Thus, for any realization of a matching as a result of (mixed strategy) equilibrium, the history independence result for the winning probability of each pairwise match in Fu, Lu, and, Pan (2015) still follows, resulting in the Hamilton-Romano totally random equilibrium.

More significantly, we extend the equivalence results in Fu, Lu, and Pan (2015) to a sequential battle-by-battle player-choice game. Here, the argument is much more involved—it is not a simple extension of sequential battles in Fu, Lu, and Pan (2015). At each subgame, the team leaders first choose players for the next battle, and then these players choose their effort levels. Thus, these players need to make a choice by foreseeing the outcomes in the subsequent subgames after the realization of the current battle's outcome. We will show, by backward induction arguments, that the team's ex ante winning probability in each subgame is the same as under the totally random matching of the remaining players by the contest organizer, thus its ex ante winning probability of the whole sequential battle-by-battle game is also the same as the ones under the Hamilton-Romano totally random Nash equilibrium in one-shot ordering choice game (Theorem 2). As a corollary, we can say that the ex ante expected equilibrium effort of each player is invariant of the type of player choice game—one-shot or sequential, since all matchings of players occur with the same ex ante probabilities in both equilibria. Thus, we add another invariance result to Fu, Lu, and Pan (2015).

In the next subsection, we provide a brief literature review. In Section 2, we will start with a three-battle contest example with exogenously …xed winning probabilities for each pairwise match between players from the two teams. This illustrates the equivalence between the outcome (ex ante team winning probability) of the one-shot game and that of the sequential move game. In Section 3, we introduce the general model using matching language and replicate Hamilton and Romano's (1998) result by using matching theory (Proposition 1). Then, in Section 4, we endogenize the winning probability of each race and show that the same results hold for both the one-shot and sequential ordering choice game (Theorems 1 and 2, and Corollary 1). In Section 5, we discuss the boundary of our result using several extensions and examples.

1.1 Related Literature

all-pay auction. They characterize the unique smnzieqfuinzsactecdinnowinu10(c)t and thus the winning chance in each battle is independent of how many games were won/lost before that battle.

payo¤ (winning probability) matrix for leader A:

Leader B

Leader A		123	132	213	231	312	321
	123						
	132						
	213						
	231						
	312						
	321						

where, for example, $= q_{11}q_{22}q_{33} + q_{11}q_{22}(1 - q_{33}) + (1 - q_{11})\, q_{22}q_{33} + q_{11}(1 - q_{22})\, q_{33}$ and ; ; ; ; are similarly de…ned. Notice that ; ; ; ; ; show up exactly once for each row and column (though some of them may take the same values).

Now, assume that leader B plays all pure strategies with probability 1

between choosing players 1, 2, or 3 in the ﬁrst round, and in the second round he chooses the rest of the orderings with probability $\frac{1}{2}$ for each (this is equivalent to choosing a player from the two remaining players with probability $\frac{1}{2}$). Clearly, leader A will place probability $\frac{1}{3}$ for each of his three players in the ﬁrst round. His equilibrium payoﬀ is again P . This discussion shows that the sequential game outcome is the same as the simultaneous game outcome. By induction, we can see that the argument works for any (odd) number of players.□

3 One-Shot Ordering Choice Game with Exogenous Winning Probabilities— the Hamilton-Romano Result

There are two teams, A and B. Each team has $2n + 1$ players where $n \in N$. The whole competition consists of $2n+1$ sequential (or simultaneous) head-to-head battles. The winning team is the one that wins $n+1$ battles. There is a team leader in charge of deciding the order

We assume that the winning probability of each match of players from teams A and B is independent of how other players are matched and which player wins. Team A's players' winning probabilities when they are matched with each of the players on team B are exogenously given by[5]

$$Q = \begin{matrix} q & \cdots & q \\ \vdots & \ddots & \vdots \\ q & \cdots & q \end{matrix}$$

where a generic match is denoted by $(i; j)$ with team A's (i's) winning probability being q . This Q matrix is perfectly general. We allow for the cases in which player i_1 does well against most of the players on team B, but i_1 somehow always loses against $j_{2\ +1}$.

The static nature of the winning probability matrix Q implies that the payoffs of this game depend only on the resulting matching, i.e., two strategy profiles that lead to the same matching will result in identical payoffs for both teams. Denote the expected payoffs from a given matching for each team by $\bar{P}\ (\ )$ (and $\bar{P}\ (\ ) = 1 - \bar{P}\ (\ )$) accordingly. Let $W = \ S \in 2^{\{1\ 2\ \ \ 2\ +1\}} : |S| \geq n + 1\ $.

$$\bar{P}\ (\ ) \equiv \ _{2} \ _{2} \ q_{(\ )} \times \ _{2} \ 1 - q_{(\ )} \ :$$

There are $(2n + 1)!$ strategy profiles $\ ;\ \ \in \ \times\ $ that achieve the same matching $\ \in M(N\ ; N\ )$, where $M(N\ ; N\ )$ denotes the set of all possible matchings. Also note that there are $(2n + 1)!$ elements in M and $((2n + 1)!)^2$ elements in $\ \times\ $. We now consider team A's winning probability when there exists a contest organizer who picks a

[5] In the next section, we endogenize winning probabilities in battles by considering a multi-battle contest game following Fu, Lu, and Pan (2015).

matching to be

$$P \equiv \frac{1}{(2n+1)!} \quad {}_{2\ (\quad)} \quad \tilde{P}\ (\ ):$$

Since the corresponding matching for any given combination of (;) is unique, we can slightly abuse the notation to let : $\times$ $\rightarrow M(N\ ;N\ )$ be the matching generated from permutations ; , such that (i) = (^{1}(i)) for all $i \in N$. Then, A's ex ante winning probability given by (;) can be written as

$$P\ (\ ;\ ) \equiv \tilde{P}\ (\ (\ ;\ )):$$

Similarly, de…ne P (;). It is clear that P (;) + P (;) = 1.

Thus, the game with two team leaders who maximize their teams' winning probability is a zero-sum game with strategy sets and , and with a $\times$ payo¤ matrix $P \equiv P\ (\ ;\ )\ _{2}\ \ _{2}$. In this case, a mixed strategy is m : □ i B for any $\in$. Therefore, we obtain the result by Hamilton and Romano (1998).

Proposition 1 (Hamilton and Romano 1998)

Q

(m ; m)

A

P ,

Note that there are many other Nash equilibria in the one-shot ordering choice game, although the equilibrium payo¤s are unique, as is shown in von Neumann (1928). For example, consider the following 2n + 1 strategies: $_1 = (i_1; ::: ; i_{2\ +1})$, $_2 = (i_{2\ +1}; i_1; ::: ; i_2\ )$, $_2 = (i_2\ ; i_{2\ +1}; i_1; ::: ; i_{2\ \ 1})$,..., and $_{2\ +1} = (i_2; ::: ; i_{2\ +1}; i_1)$. Let $\hat{m}$ be $\hat{m}\ (\ \ ) = \frac{1}{2\ +1}$ for all ` = 1; ::: ; 2n + 1 and $\hat{m}\ (\ \ ) = 0$ for any other . If team B uses strategy $\hat{m}$, then each player on team A is matched with all of the team B players with equal probability $\frac{1}{2\ +1}$. Thus, team A is indi¤erent between all strategies in . Therefore, $\hat{m}$ is one of the best responses to $\hat{m}$, and $\hat{m}\ ; \hat{m}$ is a Nash equilibrium, too. There are many other ways to select 2n + 1 behavior is exogenous. In this section, we relax this assumption following the arguments in Fu, Lu, and Pan (2015). We again assume that $N = N = 2n + 1$ and that the leaders of teams A and B simultaneously choose the player ordering at the beginning of the contest. Consider a battle between players $i \in N$ and $j \in N$. Although the same result applies to any of the examples listed in their paper, we will focus on a variation of a complete-information generalized Tullock contest (Model 6 in Fu, Lu, and Pan 2015). To apply their invariance result, assume that (ij) contest success function $q\ (x\ ; x\ )$ is (i) homogenous of degree zero in x and x, (ii) —— > 0 and

Proof. The first order conditions are

$$\frac{\partial q\ (x\ ; x\ )}{\partial x}V\ - c\ = 0 \tag{1}$$

and

$$-\frac{\partial q\ (x\ ; x\ )}{\partial x}V\ - c\ = 0 \tag{2}$$

Since $q\ (x\ ; x\ )$ is homogenous of degree zero, we have a Euler equation

$$\frac{\partial q\ (x\ ; x\ )}{\partial x}x\ + \frac{\partial q\ (x\ ; x\ )}{\partial x}x\ = 0:$$

These three equations imply

$$\frac{x}{x} = \frac{V\, c}{V\, c}:$$

Thus, team A's equilibrium winning probability is written as

$$q\ = q\ (\frac{V}{c}; \frac{V}{c}):$$

Since $q\ (x\ ; x\ )$ is homogenous of degree zero, $\frac{(\quad)}{}$ and $\frac{(\quad)}{}$ are homogeneous of degree -1. Thus, we have

$$\frac{\partial q\ (px\ ; px\ )}{\partial (px\ )} = \frac{1}{p}\frac{\partial q\ (x\ ; x\ )}{\partial x}$$

for all $p > 0$ (the same result holds for $x\ $). This implies

$$\frac{\partial q\ (px\ ; px\ )}{\partial (px\ )}pV\ - c\ = \frac{\partial q\ (x\ ; x\ )}{\partial x}V\ - c\ = 0:$$

That is, if $(x\ ; x\ ) = (x\ (i; j); x\ (i; j))$ solves the system of equations (s=0

$(x\ ;x\ ) = (px\ (i;j);px\ (i;j))$ solves the system of equations

$$\frac{@q\ \ (x\ ;x\ )}{@x}pV\ \ - c\ \ = 0$$

and

$$-\frac{@q\ \ (x\ ;x\ )}{@x}pV\ \ - c\ \ = 0:$$

We have completed the proof.□

Thus, as long as conditions (i), (ii), and (iii) are satis…ed, the winning probability of player i in a battle with player j is intact at q , since players i and j face the same probability of their battle to be pivotal p in every contest with multiple pairwise battles. This is the Observation 2 in Fu, Lu, and Pan (2015). Denote $Q(N\ ;N\ ) = (q\ )_{2\ \ \ 2}$ to be the pairwise winning probability of player i on team A against j on team B. Thus, the winning probability of team A in a multi-battle contest under …xed matching is always described by

$$\tilde{P}\ \ (\ )\equiv \sum_{2}\ \sum_{2}\ q\ _{(\ )}\times \prod_{a}\ 1-q\ _{(\ )}$$

any matching $\in M(N\ ;N\ )$, in any battle by matched players $(i;j)$ with $(i) = j$, team A wins with probability q . Thus, team A's winning probability matrix is $Q(N\ ;N\ )$. This implies that by Proposition 1, (ii) and (iii) must hold.□

4.2 The Battle-by-Battle Player Choice Game

Now, we will consider sequential battle-by-battle player-choice games. Consider a state $s \in S$ with $s = \ k;`;h;T\ ;T$ , where k is number of battles left, and ` and h denote the number of wins that teams A and B need to become the winning team at state s, respectively. Moreover, T and T denote the set of remaining players for teams A and B, respectively, and S is the set of all states. Note that $k = |T\ | = |T\ |$ and $` + h = k + 1$. We use the functions $k(s) = k$, $`(s) = `$, $h(s) = s$, $T\ (s) = T$ , and $T\ (s) = T$ to indicate the relevant information at state $s = \ k;`;h;T\ ;T$. We start with the following de…nition. In state s, let

$$P\ (s) \equiv \frac{1}{k(s)!} \sum_{2\ (\ \ (\ )\ \ (\ ))} \tilde{P}(\ ;k(s);`(s))$$

where

$$\tilde{P}(\ ;k;`) \equiv \sum_{2\ (\ )} \prod_{2} q_{\ (\ )} \times \prod_{2} 1 - q_{\ (\ )}$$

and

$$W(k;`) \equiv \ S \in 2^{f1\ \ \ g} : |S| \geq ` \ :$$

Note that $W(k;`)$ is the set of winning coalitions when a team needs to win ` out of k battles. Similar to the previous section, $P\ (s)$ is A's winning probability when there is a contest organizer who totally randomly assigns players to battles after the state s. We let $\triangle(T\ (s))$ and $\triangle(T\ (s))$ be the sets of mixed actions for leader A and B, respectively, and de…ne $\ : S \to \triangle(N\ )$ such that $\ (s) \in \triangle(T\ (s))$ as the mixed strategy of the leader v. One possible subgame perfect equilibrium strategy is $\ (s) = \frac{1}{j\ \ (\ )j}(1;1;:::;1) \in \triangle T\ (s)$ for

$v = A; B$.

In each state s, we need to consider every possible pair of players in the next battle. For i and j? The payo¤ functions of players i and j are given as

$$[q\ (x\ ; x\ )q_{0\ 0}]\,V\ - c\ x$$

and

$$[1 - q\ (x\ ; x\ )q_{0\ 0}]\,V\ - c\ x\ ;$$

respectively. The …rst order conditions are

$$\frac{@q\ (x\ ; x\ )}{@x}q_{0\ 0}V\ - c\ = 0$$

and

$$-\frac{@q\ (x\ ; x\ )}{@x}q_{0\ 0}V\ - c\ = 0:$$

Thus, $- = —$ and $q\ (-; -) = q\ $. The matrix game of this subgame is described by

		$\frac{1}{2}$	$\frac{1}{2}$
	\` = 2	jj^0	j^0j
$\frac{1}{2}$	ii^0	$q\ q_{0\ 0}$	$q\ _0q_0$
$\frac{1}{2}$	i^0i	$q\ _0q_0$	$q\ q_{0\ 0}$

Clearly, a mixed strategy pro…le with equal probability, ((s);   (s)), is an equilibrium and is unique unless $q\ q_{0\ 0}$;

members of T and T by $M(T;T)$. Similarly, denote the set of all possible matchings between the members of T and T in which player $i \in T$ is matched to player $j \in T$ by $M(T;T;(i;j))$. Then, the continuation state when player i wins is $s = (k-1; \ell - 1; h; T \setminus \{i\}; T \setminus \{j\})$ and when j wins the state is $s = (k-1; \ell; h-1; T \setminus \{i\}; T \setminus \{j\})$. We first show that (i) holds for any s with $k(s) = k$. The payoff functions of players i and j after being matched in state s are

$$u = q(x;x)P \quad s \quad V + (1-q(x;x))P \quad s \quad V -$$

$$\frac{1}{k} \quad {}_{2} \quad q \;\; P \;\; (s \quad) + (1 - q \;\;) P \;\; (s \quad)$$

$$= \frac{1}{k} \quad {}_{2} \quad q \;\; \frac{1}{(k-1)!} \quad {}_{2 \;\; (\;\; (\;\;) \;\; (\;\;))} \quad \tilde{P}(\;\; ; k-1; l-1)$$

$$+ \frac{1}{k} \quad {}_{2} \quad (1 - q \;\;) \frac{1}{(k-1)!} \quad {}_{2 \;\; (\;\; (\;\;) \;\; (\;\;))} \quad \tilde{P}(\;\; ; k-1; l)$$

$$= \frac{1}{k!} \quad {}_{2} \quad {}_{2 \;\; (\;\; (\;\;) \;\; (\;\;))} \quad q \;\; \tilde{P}(\;\; ; k-1; l-1)$$

$$+ \frac{1}{k!} \quad {}_{2} \quad {}_{2 \;\; (\;\; (\;\;) \;\; (\;\;))} \quad (1 - q \;\;) \tilde{P}(\;\; ; k-1; l)$$

$$= \frac{1}{k!} \quad {}_{2} \quad {}_{2 \;\; (\quad f \; g \quad f \; g)} \quad q \;\; \tilde{P}(\;\; ; k-1; l-1) + (1 - q \;\;) \tilde{P}(\;\; ; k-1; l)$$

$$= \quad {}_{2} \quad \frac{1}{k} \quad \frac{1}{(k-1)!} \quad {}_{2 \;\; (\quad ;(\;\;))} \quad \tilde{P}(\;\; ; k; l) \quad = P \;\; (s)$$

where $M(T \;\; ; T \;\; ; (i; j))$ is a collection of all matchings $\; : T \;\; \to T \;\;$ with $\; (i) = j$.

the same probability $\frac{1}{(\quad 1)!}$.

state s occurs with probability

$$P(s) = \quad \sim 2 \; (\; n \; (\,) \quad n \; (\,)) \; 2 \; (2 \; +1 \; (\,) \; +1 \; (\,)) \; 2 \qquad q \; \sim(\,) \times \; 1 - q \; \sim(\,) \; ;$$

player i's expected e¤ort when i is matched with j is

$$E(x \,|(i;j)) = \quad 2Sj(\;)2 \; (\,) \; (\,) \qquad P(s)p(s;(i;j))x \; (i;j)$$

$$= \quad \sim 2 \; (\; nf \, g \quad nf \, g) \; 2 \; (2 \;) \; 2 \qquad q \; \sim(\,) \times \; 1 - q \; \sim(\,) \; x \; (i;j):$$

Thus, the coe¢ cient of x (i; j) is nothing but the probability that this battle becomes pivotal. This implies that neither a sequential choice nor a one-shot choice makes a di¤erence. Hence, player i's ex ante expected e¤ort in both cases is

$$E(x) = \frac{1}{2n+1} \; 2 \quad E(x \,|(i;j))$$

$$= \frac{1}{2n+1} \; 2 \quad \sim 2 \; (\; nf \, g \quad nf \, g) \; 2f \; {}^0 2f1 \; 2 \; g{:}j \; {}^0 j = \; g \; 2 \qquad q \; \sim(\,) \times \; 1 - q \; \sim(\,) \; x \; (i;j);$$

and Fu, Lu, and Pan's (2015) total e¤ort equivalence result extends to our case, too.

Corollary 1.

Although we only considered a fully sequential player-choice game in Theorem 2, Fu, Lu, and Pan's (2015) invariance results hold even if the game involves battles with a more general temporal structure, although the argument gets messier by that (see Appendix for a formal analysis).

5 Robustness and Subtleties in Our Results

Here, we consider possible extensions of our model to see the boundaries of ou [redacted] results. It turns out that the choices of player orderings often add more subtle [redacted] results on the expected winning probability of the whole contest and ex ante e¤ [redacted]

5.1 Private Bene…ts from Winning Battles

We start with a positive result in an extension discussed in Fu, Lu, and Pan (2 [redacted] we consider the case where players get private bene…ts from winning their battle [redacted] to their team's winning the prize. Let players i and j get and from wi [redacted] (i; j). Then, players i and j's gross bene…ts $\bar{V}$ and $\bar{V}$ are written as

$$\bar{V} = \quad + p(i; j)V$$

$$\bar{V} = \quad + p(i; j)V$$

where p > 0 denotes the probability that battle (i; j) becomes pivotal. Since [redacted] equalities need to hold for any p(i; j), by Lemma 1, we can say that 79375F2211 Vi and [redacted] t 79375F22

5.2 Heterogeneous Weights

Unlike in Fu, Lu, and Pan (2015), our player-order choice game does not preserve the invariance in a team's winning probability if battles are . In the last section of Fu, Lu, and Pan (2015), they demonstrate the robustness of invariance results that allow for component battles to carry different weights. This result follows in their model, since each battle and the players who play in them are tied up together. However, in our game, team leaders assign players to each battle. If a certain battle is weighted heavily, team

5.3 Excess Players

Note that we have been assuming that the number of players who participate in the $2n + 1$ battles from each team needs to be exactly $2n + 1$. Although this assumption is natural in Fu, Lu, and Pan (2015), it is essential for our equivalence results as we can see from the following example. For simplicity, we consider a game with an exogenous winning probability matrix again.

Example 3. Suppose that there are three battles and teams A and B have four and three players, respectively. We assume the following exogenous probability matrix:

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \\ q_{41} & q_{42} & q_{43} \end{pmatrix}_A = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 \end{pmatrix} .$$

That is, player 1 on team B is a dominant player, but players 1, 2, and 3 on team A and players 2 and 3 on team B are in the exact same league. Player 4 on team A is a weak player, but is good at dealing with the dominant player 1 on team B (an assassin). In this case, if team A selects $\{1, 2, 3\}$, team A can win only when both players that are not matched with team B's dominant player win. Thus, team A's winning probability is $0.5 \times 0.5 = 0.25$. If team A includes the assassin player 4, then it has a positive winning probability only when the assassin player is matched with the dominant player. This implies that team A's winning probability is [1]

…rst round, and still has players 2, 3, and 4, while team B has players 1 and 2. Team B must win the next two races to win the team contest.

second race		$\frac{3}{4}$	$\frac{1}{4}$
	` = 1	1	2
$\frac{3}{4}$	2; 3	0:	

players win, the payo¤ matrix is:

	12	21
12	$q_{11}q_{22};\ (1-q_{11})(1-q_{22})$	$q_{12}q_{21};\ (1-q_{12})(1-q_{21})$
21	$q_{21}q_{12};\ (1-q_{21})(1-q_{12})$	$q_{22}q_{11};\ (1-q_{22})(1-q_{11})$

=

	12	21

for one star player in team A: her marginal cost is 1. Consider the case where two mediocre players were matched in battle 1 and the team A player won. Now, two team leaders are choosing which players play in the second battle. Essentially, team A's leader only has one choice: use the star player in the second battle or not. Team A needs to win only one more game, so even if it loses in the second battle, it can still win with the winning probability of the third battle. Let i_2 and j_2 be the second battle players, and i_3 and j_3 be the third battle players. Then, the second battle's stake is $1 - q$, and $x = (1 - q)\frac{}{(+)}$. Thus, team A's leader maximizes the following expected total payo¤ in this subgame.

$$
\begin{aligned}
W^{2} &= 3(q + (1 - q)q) - x - (1 - q)x \\
&= 3 \left(\frac{c}{c + c} + \frac{c}{c + c}\frac{c}{c + c} \right) - \frac{c}{c + c}\frac{c}{(c + c)^2} - \frac{c}{c + c}\frac{c}{(c + c)^2}
\end{aligned}
$$

Thus, the expected total payo¤ by setting $c = 1$ is

$$W^{2}_{=1} = 3\left(\frac{2}{3} + \frac{1}{3}\times\frac{1}{2}\right) - \frac{1}{2}\times\frac{2}{9} - \frac{1}{3}\times\frac{1}{8} = 2{:}3472$$

while the one by setting $c = 1$ is

$$W^{2}_{=1} = 3\left(\frac{1}{2} + \frac{1}{2}\times\frac{2}{3}\right) - \frac{1}{3}\times\frac{1}{8} - \frac{2}{3}\times\frac{2}{9} = 2{:}3102$$

Thus, the total randomization is not an equilibrium in this subgame. This is because if the game ends early, the third player does not need to make any e¤ort in a battle-by-battle player choice game. In contrast, in a one-shot ordering choice game, the total randomization is still a Nash equilibrium since all three games are played in a one-shot game.[8] □

[8] When a team leader maximizes the total team payo¤, the game is no longer a zero-sum game. So, there

6 Conclusions

In this paper, we show that Fu, Lu, and Pan's (2015) invariance results extend even if the team leaders strategically choose the order in which players are sent to the battlefield. The independence of each battle's winning probability extends as long as the zero homogeneity of the contest success function of each battle is satisfied. Additionally, somewhat surprisingly, the total randomization of player choice at any level is the equilibrium strategy irrespective of whether team leaders' choices are made as one-shot or battle-by-battle decisions. We also explore the robustness and limitations of our equivalence results by investigating several extensions: we found that considering ordering choice decisions add additional subtleties to the model.

Appendix

Here, we formally illustrate the way to show that Theorem 2 and Corollary 1 extend for tations of set R. For s ∈ S , let △(~(s)) be the sets of mixed strategies for leader = A; B, and de…ne : S → △(~ (s)) such that (s) ∈ (~ (s)) as the mixed strategy of the leader at state s . One possible subgame perfect equilibrium strategy is (s) = $\frac{1}{|\tilde{\ } ( )|}$(1; 1; :::; 1) ∈ △(~ (s)) for = A; B, and all t = 1; :::; t. Finally, for any action ∈ ~ (s); we denote R( ) ⊂ T (s) as the set of players involved in the action .

In order to generalize Theorem 2, we set an induction hypothesis: in each state s ∈ S , (i) subgame perfect equilibria generate each possible matching of the leftover players ∈ M (T ; T) occurs with the same probability, (ii) for all pair of players (i; j) ∈ T × T , equilibrium winning probability of i is q

A's expected winning probability in the beginning of each state $s \in S$ is $P(s)$. To show this formally, first we define $W(k;) \equiv \{S \in 2^{\{1 \quad\}} \mid |S| = \quad \}$ and

$$P(\; ; \;) = \quad q_{( )} \quad (1 - q_{( )}) \; ;$$

outan $\quad 2 \; (\quad) \quad 2 \quad 2$

which is probability of winning out of n battles given a matching . By the induction

S f o

assumptions, the expected payo¤.0569(i)6(s)-374(p)12(r)11(o)10(b)12(a)10(b)11(i)6(l)6(i)7(t)3F2211.95

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