independent, real valued, nondegenerate random variables with unknown distributions. The unknown constant $c$ is real valued, finite, and nonzero.

ASSUMPTION A3: Either the median or the mean of $U$, $V$, or $W$ is zero. The characteristic functions of $U$, $V$, and $W$ do not vanish.

Assumption A3 is mainly used for identification of the distributions of $U$, $V$, and $W$, not for the identification of $c$. Kotlarski's Lemma requires some location normalization, as in Assumption A3. Evdokimov and White (2012) provide alternative conditions under which Kotlarski's Lemma holds even when the characteristic functions of $U$, $V$, and/or $W$ can have zeros.

Kotlarski's Lemma assumes $c = 1$. We assume $c \neq 0$ because, if $c = 0$ then trivially we can only identify the distributions of $W$ and of $V + U$. Moreover, we can immediately tell if $c = 0$, because in that case the distributions of $X$ and $Y$ will be independent.

For any random variables $R$ and $S$, let $\sigma_R^2 = var(R)$ if this variance exists, and let $\sigma_{RS} = cov(R,S)$ if this covariance exists. Also let $\psi_R(t) = \ln E\left[\exp(itR)\right]$, the log characteristic function (also known as the cumulant generating function) of $R$, and similarly $\psi_{R,S}(t_1,t_2) = \ln E\left[\exp(it_1R + it_2S)\right]$.

We begin with a tiny Lemma:

LEMMA 1: Let Assumptions A1, A2, and A3 hold. If the constant $c$ is point identified, then the distributions of $U$, $V$, and $W$ are all also point identified.

equations give a bound on $c$ (it must lie between zero and the coefficient of $t_1 t_2$), but this bound is tightened below.

Lemma 1 and Theorem 1 together show how to tell if $V$ is normal or not, and show that Kotlarski's Lemma extends to point identification with an unknown factor loading $c$ as long as $V$ is non-normal.

Now consider the case where $V$ is normal. For this case, we need some more notation. For a random variable $R$, define $R$'s "largest normal factor" to be the variable $\widetilde{R}$ having the maximum variance such that $R = \widetilde{R} + \overline{R}$, where $\widetilde{R}$ and $\overline{R}$ are independently distributed and $\widetilde{R}$ is normally distributed. Without loss of generality, assume $\widetilde{R}$ has mean zero. Call $\overline{R}$ the non-normal factor. If no normal $\widetilde{R}$ exists, then $R$ does not have a normal factor, and in this case we can let $\widetilde{R} = 0$ and $\overline{R} = R$. If $R$ is normal then $\widetilde{R} = R - E[R]$ and $\overline{R} = E[R]$. See Schennach and Hu (2013) and Lewbel, Schennach, and Zhang (2020) for a similar use of normal factors. Reiersøl (1950) calls a normal factor a normal divisor.

Given a random variable $R$, the variance of $\widetilde{R}$ can be determined by

$$\sigma^2_{\widetilde{R}} = \sup\left\{\sigma^2 \geq 0 : \psi_R(t) + \tfrac{1}{2}t^2\sigma^2 \text{ is a log characteristic function}\right\}$$

If $\sigma^2_{\widetilde{R}} = 0$ then $R$ does not have a normal factor, otherwise, $\sigma^2_{\widetilde{R}}$ given by this expression is the variance of the largest normal factor $\widetilde{R}$. This follows immediately from the definition of a characteristic function, since a positive $\sigma^2_{\widetilde{R}}$ means by construction that $R$ equals the convolution of two independent random variables, one of which has the log characteristic function of a mean zero normal.[2] This means that if $R$ has a known distribution, and hence a known characteristic function, we can determine if it has a normal factor or not, and we can point identify the distributions of $\widetilde{R}$ and $\overline{R}$.

THEOREM 2: Let Assumptions A1, A2, and A3 hold. Assume $V$ is normally distributed. Then $\sigma_{\widetilde{X}\widetilde{Y}}$, $\sigma^2_{\widetilde{X}}$, and $\sigma^2_{\widetilde{Y}}$ are identified. If $\sigma_{\widetilde{X}\widetilde{Y}}\sigma\sigma^2_{\widetilde{X}} = \sigma^2_{\widetilde{Y}}\sigma\sigma_{\widetilde{X}\widetilde{Y}}$ then $c$ is point identified by $c = \sigma_{\widetilde{X}\widetilde{Y}}\sigma\sigma^2_{\widetilde{X}} = \sigma^2_{\widetilde{Y}}\sigma\sigma_{\widetilde{X}\widetilde{Y}}$ and in this case neither $W$ nor $U$ have a normal factor. Otherwise, $c$ is interval identified by $c \in \left[\sigma_{\widetilde{X}\widetilde{Y}}\sigma\sigma^2_{\widetilde{X}}, \sigma^2_{\widetilde{Y}}\sigma\sigma_{\widetilde{X}\widetilde{Y}}\right]$, and for each value of $c$ in this interval, there is a corresponding, identified unique distribution for $U$, $V$, and $W$. This interval bound on $c$ is sharp.

The fact that $c$ is point identified when neither $W$ nor $U$ have a normal factor also appears in Reiersøl (1950). The identified sets in Theorem 2 are new, but are closely related to the Frisch (1934) bounds on mismeasured linear regressions. Taken together, Lemma 1, Theorem 1, and Theorem 2 completely characterize the identification of our model.

Proof of Theorem 2: Separating $Y$ and $X$ into their normal and non-normal factors, we have $Y = \widetilde{Y} + \overline{Y}$ and $X = \widetilde{X} + \overline{X}$. Similarly, Separating $W$ and $U$ into normal and non-normal factors, we also have $Y = cV + \widetilde{W} + \overline{W}$ and $X = V + \widetilde{U} + \overline{U}$. When $V$ is normal, this implies $\widetilde{Y} = cV + \widetilde{W}$, $\overline{Y} = \overline{W}$, $\widetilde{X} = V + \widetilde{U}$ and $\overline{X} = \overline{U}$. This in turn means that, with $V$

[2] An explicit mathematical expression for "being a characteristic function" and hence defining $\sigma^2_{\widetilde{R}}$ can be obtained from Bochner's Theorem, e.g., Theorem 4.2.2 in Lukacs (1970).

normal, $\overline{X}$ and $\overline{Y}$ are independent of each other and of the joint distribution of $\widetilde{Y}$ and $\widetilde{X}$. Since the marginal distributions of $\overline{Y}$ and $\overline{X}$ are identified, we can identify the left side of

$$\Box_{Y\Box X}\Box t_1\Box t_2/ \quad \Box_{\overline{Y}}\Box t_1/ \quad \Box_{\overline{X}}\Box t_2/ \mathsf{D} \Box_{\widetilde{Y}\Box\widetilde{X}}\Box t_1\Box t_2/$$

And therefore the joint normal distribution of the mean zero variables $\widetilde{Y}$ and $\widetilde{X}$ is identified. In particular, this means that $\Box^2_{\widetilde{Y}}$, $\Box^2_{\widetilde{X}}$, and $\Box_{\widetilde{X}\widetilde{Y}}$ are identified.

The remaining step now borrows heavily from the Frisch (1934) bounds on mismeasured linear regression. From the identified second moments of $\widetilde{Y}$ and $\widetilde{X}$, we have $\Box^2_{\widetilde{Y}} \mathsf{D} c^2\Box^2_V \mathsf{C} \Box^2_{\widetilde{W}}$, $\Box^2_{\widetilde{X}} \mathsf{D} \Box^2_V \mathsf{C} \Box^2_{\widetilde{U}}$, and $\Box_{\widetilde{X}\widetilde{Y}} \mathsf{D} c\Box^2_V$, which provides three equations in the four unknown constants $\Box^2_{\widetilde{U}}$, $\Box^2_{\widetilde{W}}$, $\Box^2_V$, and $c$. The only constraints on these parameter values are that $c \mathsf{6} 0$, $\Box^2_{\widetilde{U}}$ and $\Box^2_{\widetilde{W}}$ must be non-negative (either can be zero if the corresponding normal factor doesn't exist), and $\Box^2_V$ must be positive. These being the only constraints is what makes the corresponding bounds be sharp. The equation $\Box_{\widetilde{X}\widetilde{Y}} \mathsf{D} c\Box^2_V$ means that the sign of $c$ equals the sign of $\Box_{\widetilde{X}\widetilde{Y}}$ to ensure $\Box^2_V > 0$. Then $\Box^2_{\widetilde{U}} \quad 0$ requires $\Box^2_{\widetilde{X}} \quad \Box_{\widetilde{X}\widetilde{Y}}\Box c \quad 0$ and $\Box^2_{\widetilde{W}} \quad 0$ requires $\Box^2_{\widetilde{Y}} \quad c\Box_{\widetilde{X}\widetilde{Y}} \quad 0$. Therefore, either $\Box_{\widetilde{X}\widetilde{Y}} > 0$ and $\Box_{\widetilde{X}\widetilde{Y}}\Box\Box^2_{\widetilde{X}} \quad c \quad \Box^2_{\widetilde{Y}}\Box\Box_{\widetilde{X}\widetilde{Y}}$, or $\Box_{\widetilde{X}\widetilde{Y}} < 0$ and $\Box^2_{\widetilde{Y}}\Box\Box_{\widetilde{X}\widetilde{Y}} \quad c \quad \Box_{\widetilde{X}\widetilde{Y}}\Box\Box^2_{\widetilde{X}}$. Either way $c$ lies in the interval between $\Box_{\widetilde{X}\widetilde{Y}}\Box\Box^2_{\widetilde{X}}$ and $\Box^2_{\widetilde{Y}}\Box\Box_{\widetilde{X}\widetilde{Y}}$,

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